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Formfactor perturbation expansions and confinement in the Ising field theory

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Abstract
We study the particle spectrum $M_n(h)$ in the two-dimensional ferromagnetic Ising field theory in a weak external magnetic field h. According to the Wu and McCoy scenario of the weak confinement, pairs of fermions (domain walls) are coupled into bosonic kink–antikink bound states at small $h > 0$. Fluctuations with more than two fermions also contribute to the wavefunctions of the compound particles, leading to multi-fermion corrections to their masses $M_n(h)$ in higher orders in h. We describe a perturbative procedure, which allows us to account for both multi-fermion fluctuations and the long-range confining interaction between fermions, and leads to the formfactor expansions for the renormalized parameters of the model. We obtain integral representations for the third-order multi-fermion correction to the mass $M_n(h)$, which arises from the regular correction to the kernel of the Bethe–Salpeter equation.

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1. Introduction

In recent years, much progress has been achieved in the understanding of the scaling limit of the two-dimensional Ising model, which is known as the Ising field theory (IFT); for a review, see [1]. Providing direct information about the Ising universality class in two dimensions, the IFT can also be viewed as a continuous dynamical model of the one-dimensional uniaxial ferromagnet. Being, perhaps, the simplest relativistic model describing confinement of topological excitations, the IFT can give a deep insight into some nontrivial aspects of confinement in particle and condensed matter physics.

The IFT contains parameters m and h, which are proportional to the deviations of temperature T and magnetic field H from their critical values in the two-dimensional lattice Ising model, $m \sim (T_c - T), h \sim H$. At the critical point $m = 0, h = 0$, the IFT reduces to the conformal field theory (CFT) with central charge $c = \frac{1}{2}$, whose Euclidean action $\mathcal{A}_{\mathrm{CFT}}$

describes free massless Majorana fermions. It has two relevant operators: the energy density $\varepsilon(x)$ and the order spin operator $\sigma(x)$. The IFT can be defined as the perturbation of the Ising conformal field theory by these two operators, which is described by the action [2]

$$\mathcal{A}_{\rm IFT} = \mathcal{A}_{\rm CFT} + 2\pi m \int \varepsilon(x)\,\mathrm{d}^2x - h \int \sigma(x)\,\mathrm{d}^2x. \qquad (1)$$

In fact, only one dimensionless parameter $\eta = m/|h|^{8/15}$ determines the physics of the IFT.

The IFT, being not integrable for generic h and m, admits exact solutions along the directions $h = 0$ and $m = 0$. The line $h = 0$, $m \neq 0$ corresponds to Onsager's exact solution [3]. Fermions remain free here, but gain the mass $|m|$. In the disordered (paramagnetic) phase $m < 0$ these fermions are ordinary particles, while in the ordered (ferromagnetic) phase $m > 0$ they are interpreted as topological excitations (kinks), which separate regions with oppositely directed spontaneous magnetization. A nonzero magnetic field $h > 0$ induces interaction between fermions, breaking the integrability of the IFT at $m \neq 0$. On the other hand, the IFT has a remarkable exact solution at $m = 0$, $h \neq 0$ containing eight massive particles, which was found by Zamolodchikov [4].

Beyond the integrable directions, the IFT can be studied by approximate methods—numerical and analytical. An effective numerical method known as the truncated conformal space approach was discovered by Yurov and Alexei Zamolodchikov [5, 6]. Fonseca and Zamolodchikov [7] modified this technique and applied it to the analysis of analytical properties of the IFT free energy continued to complex values of the scaling parameter η.

For an analytical study of the IFT for h and m close to the integrable directions, it is natural to exploit perturbation expansions. Formfactor perturbation theory developed by Delfino *et al* [8] has been applied [8, 9] to calculate the variation of the particle mass spectrum and the decay widths of a non-stable particle for small η, i.e. near the line $m = 0$. Further information on the resonance parameters in the IFT at small η was extracted from the finite volume data by Pozsgay and Takács [10]. One could expect that the perturbation expansion at $m \neq 0$ and small h should be simpler, since the IFT is free at $h = 0$ and $m \neq 0$. Though this is really the case in the high-temperature phase $m < 0$, the small-h expansion at $m > 0$ turns out to be rather non-trivial due to the long-range attractive potential between the neighbouring fermions, which is induced by the external magnetic field $h > 0$. This attractive interaction cannot be accounted for by the straightforward formfactor perturbation theory at small values of h, and leads to confinement of fermions.

The effect of a small magnetic field h, which breaks the $\mathbb{Z}_2$-symmetry in the ordered phase $m > 0$ in the IFT, can be qualitatively understood by the following simple arguments first developed by McCoy and Wu [11]. At $h = 0$, two ferromagnetic ground states $|0_+\rangle$ and $|0_-\rangle$ with spontaneous magnetizations $+\bar{\sigma}$ and $-\bar{\sigma}$ have the same energy. A weak magnetic field $h > 0$ removes degeneration decreasing the energy of the state $|0_+\rangle$ and increasing the energy of the state $|0_-\rangle$, which becomes metastable. In order to generate a domain of the metastable phase in the stable surrounding, one needs to add the energy proportional to the length of the domain. In other words, two domain walls bounding such a domain attract one another with the energy $2h\bar{\sigma}l$ proportional to their separation l; see figure 1. The long-range attraction leads to confinement: all domain walls are coupled into pairs at arbitrary small $h > 0$. Elementary excitations are now the domains bounded by two kinks, while an isolated kink gains infinite energy.

The mechanism of confinement outlined above is quite general in one-dimensional systems. It is realized in continuous one-dimensional models, such as the multi-frequency sinh-Gordon model [12], q-state Potts field theory [13], and in the discrete Ising spin chain [14]. Confinement of topological excitations in the one-dimensional antiferromagnet has been

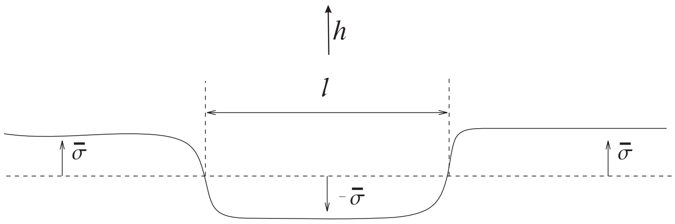


Figure 1. Two kinks interact with the energy $2h\bar{\sigma}l$.

observed experimentally by Kenzelmann *et al* [15]. On the other hand, there are a lot of similarities between confinement in the IFT and in the 't Hooft's model for two-dimensional multicolour QCD [16]; see the discussion in [2]. Accordingly, the fermions and their bound states in the IFT in the confinement regime are called 'quarks' and 'mesons', respectively.

At small h, the weak confinement regime is realized in the IFT. In this regime, the mass spectrum M_n of mesons is dense in the segment $[2m, \infty)$. Two asymptotic expansions describe M_n at $h \to 0$ in different regions of this segment. Near the edge point $2m$ (i.e. for fixed n at $h \to 0$), one can use the *low energy expansion* in fractional powers of the magnetic field [2, 7]. On the other hand, for $n \gg 1$ and $h \to 0$, the *semiclassical expansion* in integer powers of h can be applied [2, 17]. The derivation of both expansions is based on the perturbative analysis of the Bethe–Salpeter equation, which determines the meson mass and wavefunction in the *two-quark approximation*. The latter implies that one approximates the meson wavefunction (the eigenstate of the IFT Hamiltonian) by the two-quark state, neglecting multi-quark (four-quark, six-quark, etc) contributions to it. The two-quark approximation is asymptotically exact in the limit $h \to 0$ giving correct meson masses in the leading order in h. However, starting from the second order in h, it is necessary to take into account the virtual multi-quark fluctuations. Note that multi-quark effects are also essential for interesting phenomena such as the decay of unstable mesons and inelastic meson scattering.

The second-order multi-quark correction to the meson mass was obtained by Fonseca and Zamolodchikov [2]. These authors also demonstrated that the multi-quark corrections could come up in the weak-coupling expansions of the meson masses M_n in three ways:

(i) through the radiative corrections of the quark mass and self-energy,
(ii) by renormalization of the long-range attractive force between the neighbouring quarks (the 'string-tension'),
(iii) by modifying the regular part of the Bethe–Salpeter kernel, which is responsible for the pair interaction between quarks at short distances.

It turns out that only the first contribution (i) gives rise to the second-order correction to the meson mass, while (ii) and (iii) should show up only in the third-order correction, which is still unknown.

The extension of the weak-coupling expansions for the meson masses to the third order in the magnetic field presents an interesting problem, which we address in this work. It could give us some insight into the role of the multi-particle fluctuations in the composite particles in non-integrable models exhibiting confinement. Since the multi-quark effect is also responsible for the decay of unstable mesons, this should manifest itself in some form in the perturbative meson mass spectrum near and above the stability threshold. Note that an accurate numerical calculation of the lowest meson masses was reported in [2], which clearly indicates the contribution of the multi-quark fluctuations.

Since the problem outlined above is rather involved, here we shall concentrate only on three parts of it. First, we extend the semiclassical expansion of the original (written in the two-quark approximation) Bethe–Salpeter equation to the third order in h. Second, describe the formfactor perturbative technique, which is suitable to deal with multi-particle fluctuations in systems with confinement. Finally, we obtain the integral representations for the 'local' multi-quark correction of the meson masses, i.e. corrections (iii) induced by renormalization of the *local* interaction between quarks.

The paper is organized as follows. In section 2, we describe the definition of the IFT and its operator content. In two subsequent sections, we briefly summarize the recent progress in the theory of the weak confinement in the IFT. Section 3 introduces the Bethe–Salpeter equation and its weak-coupling expansions and section 4 contains preliminary discussion of the multi-quark corrections to the meson masses. In section 5 we develop a formfactor perturbative procedure, which is modified to a system with a long-range confining interaction between fermions. It is based on the partial diagonalization of the Hamiltonian in the fermionic number and allows one to effectively account for the multi-quark fluctuations by 'dressing' the fermionic operators. In section 6, we describe a compact integral representation for the local third-order correction to the meson mass, which is further analysed in appendix B. Appendix A contains a perturbative solution of the 'bare' Bethe–Salpeter equation to the third order in h. Concluding remarks are presented in section 7.

2. The model

The Ising field theory is the Euclidean field theory, which describes the scaling limit of the two-dimensional lattice Ising model in the critical region $T \to T_c$, $H \to 0$. It is defined by the action

$$\mathcal{A}_{\rm IFT} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\psi\bar\partial\psi + \overline{\psi}\partial\overline{\psi} + {\rm i}m\overline{\psi}\psi]\,{\rm d}^2x - h\int_{-\infty}^{\infty} {\rm d}^2x\,\sigma(x). \tag{2}$$

Here, x denotes a point in the plane $\mathbb{R}^2$ with Cartesian coordinates $\langle {\rm x}(x), {\rm y}(x)\rangle$ and the complex coordinate $z = {\rm x}+{\rm iy}$, $\partial = \frac{1}{2}(\partial_{\rm x} - {\rm i}\partial_{\rm y})$, $\bar\partial = \frac{1}{2}(\partial_{\rm x} + {\rm i}\partial_{\rm y})$. Action (2) is covariant under rotation, and becomes Lorentz covariant after the Wick turn ${\rm y} \to {\rm i}t$.

Corresponding to action (2), the Hamiltonian can be written in the form

$$\begin{aligned}
&\mathcal{H} = \mathcal{H}_0 + hV, \qquad \text{where}\\
&\mathcal{H}_0 = \int_{-\infty}^{\infty}\frac{{\rm d}p}{2\pi}\omega(p)\mathbf{a}^\dagger(p)\mathbf{a}(p),\\
&V = -\int_{-\infty}^{\infty} {\rm dx}\,\sigma({\rm x}),
\end{aligned} \tag{3}$$

and $\omega(p) = (p^2+m^2)^{1/2}$ is the spectrum of free fermions. Fermionic operators $\mathbf{a}^\dagger(p')$ and $\mathbf{a}(p)$ obey the canonical anticommutational relations

$$\{\mathbf{a}(p), \mathbf{a}^\dagger(p')\} = 2\pi\delta(p-p'), \qquad \{\mathbf{a}(p), \mathbf{a}(p')\} = \{\mathbf{a}^\dagger(p), \mathbf{a}^\dagger(p')\} = 0.$$

Commonly used are also fermionic operators $a(\beta), a^\dagger(\beta)$, corresponding to the rapidity variable $\beta = {\rm arcsinh}(p/m)$:

$$a(\beta) = \omega(p)^{1/2}\mathbf{a}(p), \qquad a^\dagger(\beta) = \omega(p)^{1/2}\mathbf{a}^\dagger(p).$$

Notations

$$|p_1,\ldots,p_N\rangle = \mathbf{a}^\dagger(p_1)\cdots\mathbf{a}^\dagger(p_N)|0\rangle, \qquad \langle p_1,\ldots,p_N| = \langle 0|\mathbf{a}(p_1)\cdots\mathbf{a}(p_N),$$
$$|\beta_1,\ldots,\beta_N\rangle = a^\dagger(\beta_1)\cdots a^\dagger(\beta_N|0\rangle, \qquad \langle\beta_1,\ldots,\beta_N| = \langle 0|a(\beta_1)\cdots a(\beta_N)$$

for the fermionic basis states with definite momenta will be used.

The order spin operator $\sigma(\mathrm{x}) = \sigma(\mathrm{x}, \mathrm{y})|_{\mathrm{y}=0}$ in the ordered phase $T < T_C$ (i.e. at $m > 0$) can be determined in the infinite line $\mathrm{x} \in \mathbb{R}$ as the normally ordered exponent [18, 19]:

$$\begin{aligned} &\sigma(\mathrm{x}) = \bar{\sigma} : \mathrm{e}^{\rho(\mathrm{x})/2} :, \\ &\frac{\rho(\mathrm{x})}{2} = \int_{\mathrm{x}}^{\infty} \mathrm{d}\mathrm{x}' (\chi(\mathrm{x}', \mathrm{y}) \partial_{\mathrm{y}} \chi(\mathrm{x}', \mathrm{y}))|_{\mathrm{y}=0}, \\ &\chi(\mathrm{x}, \mathrm{y}) = \mathrm{i} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{\mathrm{i}p\mathrm{x}}}{\sqrt{\omega(p)}} (\mathbf{a}^{\dagger}(-p)\, \mathrm{e}^{\omega(p)\mathrm{y}} - \mathbf{a}(p)\, \mathrm{e}^{-\omega(p)\mathrm{y}}), \end{aligned} \tag{4}$$

where $\bar{\sigma} = m^{1/8} 2^{1/12} e^{-1/8} A^{3/2}$ is the zero-field vacuum expectation value of the order field (spontaneous magnetization) and $A = 1.282\,43\ldots$ is Glaisher's constant.

Alternatively, operators $\sigma(\mathrm{x})$ can be completely characterized by their formfactors $\langle \beta_1, \ldots, \beta_K | \sigma(0) | \beta_1', \ldots, \beta_N' \rangle$, whose explicit expressions are well known [7, 20]. In the ordered phase

$$\begin{aligned} \langle \beta_1, \ldots, \beta_K | \sigma(0) | \beta_1', \ldots, \beta_N' \rangle &= \mathrm{i}^{(K+N)/2} \bar{\sigma} \prod_{0<i<j\leqslant K} \tanh\left(\frac{\beta_i - \beta_j}{2}\right) \\ &\times \prod_{0<k<q\leqslant N} \tanh\left(\frac{\beta_k' - \beta_q'}{2}\right) \prod_{\substack{0<s\leqslant K \\ 0<t\leqslant N}} \coth\left(\frac{\beta_s - \beta_t'}{2}\right), \end{aligned} \tag{5}$$

if $(K + N)$ is even and $\langle \beta_1, \ldots, \beta_K | \sigma(0) | \beta_1', \ldots, \beta_N' \rangle = 0$ for odd $(K + N)$. The right-hand side in (5) contains factors $\coth\left(\frac{\beta_s - \beta_t'}{2}\right)$, which are singular at $\beta_s = \beta_t'$. These kinematic singularities should be understood in the sense of the Cauchy principal value

$$\coth\left(\frac{\beta_s - \beta_t'}{2}\right) \to \frac{1}{4}\left[\coth\left(\frac{\beta_s - \beta_t' + \mathrm{i}0}{2}\right) + \coth\left(\frac{\beta_s - \beta_t' - \mathrm{i}0}{2}\right)\right].$$

Note that the Wick expansion holds for formfactors (5) of the spin operator. For example,

$$\begin{aligned} \bar{\sigma} \langle \beta_1, \beta_2 | \sigma(0) | \beta_1', \beta_2' \rangle &= \langle \beta_1 | \sigma(0) | \beta_2' \rangle \langle \beta_2 | \sigma(0) | \beta_1' \rangle \\ &- \langle \beta_1 | \sigma(0) | \beta_1' \rangle \langle \beta_2 | \sigma(0) | \beta_2' \rangle + \langle \beta_1, \beta_2 | \sigma(0) | 0 \rangle \langle 0 | \sigma(0) | \beta_1', \beta_2' \rangle. \end{aligned}$$

3. Bethe–Salpeter equation

The meson energy spectra $\Delta E_n(P)$ can be formally determined from the eigenvalue problem:

$$\mathcal{H} | \Phi_n(P) \rangle = [\Delta E_n(P) + E_{\mathrm{vac}}] | \Phi_n(P) \rangle, \tag{6}$$

$$\mathcal{H} = \mathcal{H}_0 + hV,$$

$$\hat{P} | \Phi_n(P) \rangle = P | \Phi_n(P) \rangle, \tag{7}$$

where $\hat{P}$ is the total momentum operator:

$$\hat{P} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} p \mathbf{a}^{\dagger}(p) \mathbf{a}(p)$$

and E_{vac} is the ground-state energy, which is proportional to the length of the system L.

The eigenvalue problem (6) is quite difficult, since the Hamiltonian contains the order spin operator $\sigma(\mathrm{x})$, which is highly nonlinear in fermionic fields. A significant simplification can be provided by the two-quark approximation [7, 21]. It implies that one replaces the exact Hamiltonian eigenvalue problem (6), (7) by its projection to the two-quark subspace $\mathbb{F}_2$ of the

Fock space $\mathbb{F}$:

$$\begin{aligned} &\mathcal{P}_2\mathcal{H}|\widetilde{\Phi}_n(p)\rangle = [\Delta\widetilde{E}_n(P) + \widetilde{E}_{\text{vac}}]|\widetilde{\Phi}_n(P)\rangle, \\ &\hat{P}|\widetilde{\Phi}_n(P)\rangle = P|\widetilde{\Phi}_n(P)\rangle, \\ &|\widetilde{\Phi}_n(P)\rangle \in \mathbb{F}_2. \end{aligned} \tag{8}$$

Here, $\mathcal{P}_n$ denotes the orthogonal projector onto the n-quark subspace $\mathbb{F}_n$ of $\mathbb{F}$. Tildes distinguish solutions of (8) from those of the exact eigenvalue problem (6).

In the momentum representation, equation (8) takes the form [2]

$$[\omega(P/2+p) + \omega(P/2-p) - \Delta\widetilde{E}(P)]\Psi_P(p) = f_0 \fint_{-\infty}^{\infty} G_P(p|k)\Psi_P(k)\frac{\mathrm{d}k}{2\pi}, \tag{9}$$

where $\fint$ denotes the Cauchy principal value integral:

$$\begin{aligned} &\langle P'/2 - p, P'/2 + p|\widetilde{\Phi}(P)\rangle = 2\pi\delta(P' - P)\Psi_P(p), \\ &G_P(p|k) = \mathcal{G}(P/2+p, P/2-p|P/2+k, P/2-k), \end{aligned} \tag{10}$$

$$\begin{aligned} \mathcal{G}(p_1, p_2|k_1, k_2) = \frac{1}{4\bar{\sigma}}\langle p_2, p_1|\sigma(0)|k_1, k_2\rangle = \frac{1/4}{[\omega(p_1)\omega(p_2)\omega(k_1)\omega(k_2)]^{1/2}} \\ \cdot\left[\frac{\omega(p_1)+\omega(k_2)}{p_1-k_2}\frac{\omega(p_2)+\omega(k_1)}{p_2-k_1} - \frac{\omega(p_1)+\omega(k_1)}{p_1-k_1}\frac{\omega(p_2)+\omega(k_2)}{p_2-k_2}\right. \\ \left. + \frac{p_1-p_2}{\omega(p_1)+\omega(p_2)}\frac{k_1-k_2}{\omega(k_1)+\omega(k_2)}\right], \end{aligned} \tag{11}$$

and $f_0 = 2h\bar{\sigma} = \lambda m^2$ is the 'bare string tension'. Index n is omitted in (9), (10). Note that $\Psi_P(p)$ is an odd function of p, and

$$G_P(p|k) = \frac{1}{(p-k)^2} - \frac{1}{(p+k)^2} + G_P^{(\text{reg})}(p|k),$$

where $G_P^{(\text{reg})}(p|k)$ is regular at real p and q. The pole terms in $G_P(p|k)$ produce after the Fourier transform the long-range linear attractive potential $f_0|\mathrm{x}|$ proportional to the distance $|\mathrm{x}|$ between the two quarks. The regular term $G_P^{(\text{reg})}(p|k)$ is responsible for the local interaction between quarks vanishing at distances $\gg m^{-1}$.

Equation (9) is the Bethe–Salpeter equation written in a generic momentum frame. It simplifies in two cases.

- In the frame of the centre of mass of two quarks [7], $P = p_1 + p_2 = 0$:

$$\begin{aligned} [2\omega(p) - \Delta\widetilde{E}(0)]\Psi_0(p) = f_0 \fint_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\Psi_0(k)}{2\omega(p)\omega(k)} \\ \cdot\left[\left(\frac{\omega(p)+\omega(k)}{p-k}\right)^2 + \frac{1}{2}\frac{pk}{\omega(p)\omega(k)}\right]. \end{aligned} \tag{12}$$

- In the infinite momentum frame (see appendix A in [2]), $P \to \infty$:

$$\left[\frac{m^2}{1-u^2} - \frac{\widetilde{M}^2}{4}\right]\Phi(u) = f_0 \fint_{-1}^{1} F(u|v)\Phi(v)\frac{\mathrm{d}v}{2\pi}, \tag{13}$$

where the scaled variables $u = (p_1 - p_2)/P$ and $v = (q_1 - q_2)/P$ have been used, and

$$F(u|v) = [(1-u^2)(1-v^2)]^{-1/2}\left[\frac{1-uv}{(u-v)^2} - \frac{1+uv}{(u+v)^2} + \frac{uv}{4}\right],$$

$$\Phi(u) = \lim_{P\to\infty}\Psi_P(Pu/2).$$

The following large-P asymptotic behaviour of $\Delta\widetilde{E}(P)$ was assumed in [2] in deriving (13) from (9):

$$\Delta\widetilde{E}(P) = |P| + \frac{\widetilde{M}^2}{2|P|} + O(|P|^{-3}). \tag{14}$$

The Bethe–Salpeter equation (9) and its particular cases (12), (13) are the linear singular integral equations [22]. Different techniques [2, 7, 17] have been developed for their perturbative solutions in the weak-coupling limit $\lambda \to 0$. Fonseca and Zamolodchikov calculated [2] several initial terms in the low-energy expansion (for fixed n and $\lambda \to 0$) for the eigenvalues of equation (13):

$$\frac{\widetilde{M}_n^2}{4m^2} - 1 = z_n t^2 + \frac{z_n^2}{5}t^4 - \left(\frac{3z_n^3}{175} + \frac{57}{280}\right)t^6 + \left(\frac{23z_n^4}{7875} + \frac{1543z_n}{12\,600}\right)t^8 + \frac{13}{1120\pi}t^9 + \left(-\frac{1894z_n^5}{3031\,875} - \frac{23\,983z_n^2}{242\,550}\right)t^{10} + \frac{3313z_n}{10\,080\pi}t^{11} + \dots, \tag{15}$$

where $t = \lambda^{1/3}$ and $(-z_n)$ is the zero of the Airy function, $\mathrm{Ai}(-z_n) = 0$. The leading term in the above expansion reproduces the old result of McCoy and Wu [11].

To the second order in λ, semiclassical expansions (for $n \gg 1$ and $\lambda \to 0$) for $\widetilde{M}^2$ and for $\Delta\widetilde{E}(0)$ were found in [2, 17], respectively. We extend the former expansion to the third order in λ using the technique, which was previously applied in a similar discrete-chain problem [14]. This calculation is described in appendix A, and the result reads as

$$\frac{\widetilde{M}_n^2}{4m^2} = \cosh^2\theta_n, \tag{16}$$

where θ_n solves the equation

$$\sinh 2\theta_n - 2\theta_n = 2\pi\lambda(n - 1/4) + 2\lambda^2 S_1(\theta_n) + 2\lambda^3 S_2(\theta_n) + O(\lambda^4), \tag{17}$$

and

$$S_1(\theta_n) = \frac{1}{\sinh(2\theta_n)}\left(\frac{5}{24\sinh^2\theta_n} - \frac{1}{12} + \frac{1}{4\cosh^2\theta_n} - \frac{\sinh^2\theta_n}{6}\right), \tag{18}$$

$$S_2(\theta_n) = \frac{1}{192\pi\sinh^6(2\theta)}\{-999\theta_n - 3\theta_n[648\cosh(2\theta_n) + 228\cosh(4\theta_n) + 56\cosh(6\theta_n) + 15\cosh(8\theta_n)] + 546\sinh(2\theta_n) + 363\sinh(4\theta_n) + 170\sinh(6\theta_n) + 33\sinh(8\theta_n) + \sinh(12\theta_n)\}. \tag{19}$$

To the second order, (16)–(19) agree with [2].

4. Beyond the two-quark approximation

Eigenvalues $\Delta\widetilde{E}_n(P)$ of the Bethe–Salpeter equation (9) are not the same as the eigenvalues $\Delta E_n(P)$ of the initial problem (6):

$$\Delta E_n(P) = \Delta\widetilde{E}_n(P) + \delta E_n(P). \tag{20}$$

The difference $\delta E_n(P)$ is caused by the multi-quark corrections, which are ignored in (9), but contribute to $\Delta E_n(P)$. The exact meson energy spectra should have the form

$$\Delta E_n(P) = \left(M_n^2 + P^2\right)^{1/2} \tag{21}$$

due to the Lorentz covariance of the IFT, but this form does not hold [2] for the meson energies $\Delta\widetilde{E}_n(P)$ determined in the two-quark approximation.

In the $P \to \infty$ limit, equation (20) yields due to (14) and (21)

$$M_n^2 = \widetilde{M}_n^2 + \delta M_n^2,$$

where

$$\delta M_n^2 = \lim_{P\to\infty} [2P\delta E_n(P)].$$

The first analysis of the multi-quark corrections to the meson masses has been done by Fonseca and Zamolodchikov [2]. They claim that multi-quark corrections treated perturbatively in λ should modify the Bethe–Salpeter equation (9) to the form

$$[\varepsilon(P/2+p)+\varepsilon(P/2-p)-\Delta E(P)]\,\Psi_P(p) = f \,⨍_{-\infty}^{\infty} \mathbb{G}_P(p|k)\Psi_P(k)\frac{\mathrm{d}k}{2\pi}. \tag{22}$$

Here $\varepsilon(p)$ and f are the renormalized quark dispersion law and the renormalized string tension, respectively. The renormalized kernel $\mathbb{G}_P(p|k)$ is assumed to have the structure

$$\mathbb{G}_P(p|k) = G_P(p|k) + \Delta\mathbb{G}_P^{(\mathrm{reg})}(p|k), \tag{23}$$

where $G_P(p|k)$ is the original kernel (11), and the correction term $\Delta\mathbb{G}_P^{(\mathrm{reg})}(p|k) = O(\lambda)$, being regular at $k = \pm p$, effectively modifies the pair interaction between quarks at short distances $\lesssim m^{-1}$.

Note that the renormalized quark energy does not have the Lorentz covariant form [2]

$$\varepsilon(p) = (p^2+m^2)^{1/2} + \delta\varepsilon(p) = \left(p^2+m_q^2\right)^{1/2} + \Delta\varepsilon(p),$$

since quarks are not free particles at $h > 0$ due to their confinement. Assuming $\Delta\varepsilon(p) = O(|p|^{-3})$ at $p \to \infty$, one can define the 'dressed' quark mass m_q from the large-p asymptotics of $\varepsilon(p)$:

$$\varepsilon(p) = |p| + \frac{m_q^2}{2|p|} + O(|p|^{-3}).$$

There are no nonperturbative definitions of renormalized quantities in equation (22). Instead, it is expected that they can be determined order by order by their power series in λ:

$$m_q^2 = m^2(1 + a_2\lambda^2 + a_3\lambda^3 + \cdots), \tag{24a}$$

$$\delta\varepsilon(p) = \delta_2\varepsilon(p) + \delta_3\varepsilon(p) + O(\lambda^4), \tag{24b}$$

$$f = f_0(1 + c_2\lambda^2 + c_4\lambda^4 + \cdots) \tag{24c}$$

$$\Delta\mathbb{G}_P^{(\mathrm{reg})}(p|k) = \Delta_1\mathbb{G}_P^{(\mathrm{reg})}(p|k) + \Delta_2\mathbb{G}_P^{(\mathrm{reg})}(p|k) + O(\lambda^3). \tag{24d}$$

Let us briefly summarize what is known about the coefficients in the above expansions. Fonseca and Zamolodchikov [23] analysed the exact integral representation for the coefficient a_2 in (24a), and obtained from it the value

$$a_2 = 0.071\,010\,809\ldots. \tag{25}$$

On the other hand, one can expand a_2 to the sum

$$a_2 = a_{2,3} + a_{2,5} + \cdots$$

of the second-order (in λ) diagrams with three, five, etc, quarks in the intermediate state. The contribution of three-quark diagrams to a_2 was estimated in [7]:

$$a_{2,3} \approx 0.07\ldots \tag{26}$$

We obtain its exact value

$$a_{2,3} = \frac{1}{16} + \frac{1}{12\pi^2} = 0.070\,94\ldots; \tag{27}$$

this calculation will be presented elsewhere. Comparison of (27) with (25) shows that the second-order radiative correction to the quark mass is essentially determined by the three-quark contribution. Diagrams with five and more quarks in the intermediate state give less than 0.1% of a_2.

The term of order λ^2 in expansion (24*b*) for $\delta\varepsilon(p)$ was found by Fonseca and Zamolodchikov [2]:

$$\delta_2\varepsilon(p) = \frac{\lambda^2}{2}\frac{m^2 a_2}{\omega(p)} - \frac{\lambda^2}{8}\frac{m^4 p^2}{\omega^5(p)}. \tag{28}$$

They have also given strong arguments that coefficients c_{2k} in expansion (24*c*) should be simply related to coefficients $\tilde{g}_j$ in the well-known weak-h expansion [20] for the vacuum energy E_{vac}:

$$E_{\text{vac}} = Lm^2\big(-\tfrac{1}{2}\lambda + \tilde{g}_2\lambda^2 + \tilde{g}_3\lambda^3 + \tilde{g}_4\lambda^4 + \cdots\big), \tag{29}$$

namely

$$c_{2k} = -2\tilde{g}_{2k+1}. \tag{30}$$

In particular, $c_2 = -0.003\,889\ldots$.

It is not difficult to modify the weak-coupling expansions (both low-energy and semiclassical) to account for renormalized quantities in the Bethe–Salpeter equation (22) and to express multi-quark correction δM_n in terms of coefficients in (24*a*)–(24*d*). It turns out [2] that for calculation of the meson masses M_n to the third order in λ, it would be sufficient to know the renormalized quark mass m_q and the string tension f to the third order in λ, and the 'regular' term $\Delta\mathbb{G}_P^{(\text{reg})}(p|q)$ in (23) to the linear order in λ in the limit $P \to \infty$. To this end, one needs to determine two unknown quantities: the third-order correction to the quark mass (coefficient a_3 in (24*a*)) and the kernel $\Delta_1\mathbb{G}_\infty^{(\text{reg})}(p|k)$ in (24*d*). In fact, we need only the diagonal part of the latter, $\Delta_1\mathbb{G}_\infty^{(\text{reg})}(p|p)$.

The problem of explicit calculation of a_3 and $\Delta_1\mathbb{G}_P^{(\text{reg})}(p|k)$ is quite difficult. Here, we do not try to find its complete solution. Instead, in subsequent sections we shall obtain several representations for these quantities in terms of formfactors of spin operators $\sigma(x)$ and their products $\sigma(x_1)\sigma(x_2)$.

5. Diagonalization of the Hamiltonian in the fermionic number

The Bethe–Salpeter equation (9) is approximate, since the IFT Hamiltonian (3) does not conserve the number of fermions—the 'bare' quarks. Let us try to find a unitary operator $U(h)$, which transfers operators generating 'bare' fermions to operators generating 'dressed' fermions such that their number would be conserved by the evolution operator. It is clear that the two-fermion Bethe–Salpeter equation, written for these 'dressed' fermions, should be exact, and it could be identified with the renormalized Bethe–Salpeter equation (22).

Let $\underline{\mathbf{a}}^\dagger(p), \underline{\mathbf{a}}(p)$ be a set of creation/annihilation operators of the 'dressed' fermions, which are related to the 'bare' ones by the unitary transform

$$\mathbf{a}(p) = U(h)\underline{\mathbf{a}}(p)U(h)^{-1}, \qquad \mathbf{a}^\dagger(p) = U(h)\underline{\mathbf{a}}^\dagger(p)U(h)^{-1}$$

with the operator $U(h)$ depending on the magnetic field h. We shall also underline all 'dressed' operators and states:

$$\underline{A} = U(h)^{-1}AU(h), \qquad |\underline{\Phi}\rangle = U(h)^{-1}|\Phi\rangle.$$

Expanding $U(h)$ to the power series in h:

$$U(h) = 1 + \sum_{n=1}^{\infty} h^n \mathcal{F}_n,$$

we obtain a set of equalities following from the unitarity condition $U(h)U(h)^\dagger = 1$:

$$\mathcal{F}_1 + \mathcal{F}_1^\dagger = 0,$$
$$\mathcal{F}_2 = \frac{\mathcal{F}_1^2}{2} + \Lambda, \qquad \Lambda^\dagger = -\Lambda,$$
$$\mathcal{F}_3 = \frac{\Lambda\mathcal{F}_1 + \mathcal{F}_1\Lambda}{2} + Y, \qquad Y^\dagger = -Y,$$
$$\ldots$$

Denote by $\underline{N}$ the operator of the number of 'dressed' fermions

$$\underline{N} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \underline{\mathbf{a}}^\dagger(p)\underline{\mathbf{a}}(p),$$

and by $\underline{\mathcal{P}}_n$ the projector operators onto the subspaces of n 'dressed' fermions. For an operator A acting in the Fock space, let us separate the diagonal and off-diagonal parts in the 'dressed' fermion number n, $A = A_d + A_s$, where

$$A_d = \sum_{n=0}^{\infty} \underline{\mathcal{P}}_n A \underline{\mathcal{P}}_n \qquad \text{and} \qquad A_s = A - A_d.$$

We require that $\mathcal{H}$ and $\underline{N}$ commute, $[\mathcal{H}, \underline{N}] = 0$, or, equivalently,

$$\mathcal{H}_s = 0. \tag{31}$$

Rewriting (31) as

$$\mathcal{H}_s = (U(h)\underline{\mathcal{H}}U(h)^{-1})_s = 0,$$

one obtains

$$\left((1 + h\mathcal{F}_1 + h^2\mathcal{F}_2 + h^3\mathcal{F}_3 + \cdots)(\underline{\mathcal{H}}_0 + h\underline{V})\left(1 + h\mathcal{F}_1^\dagger + h^2\mathcal{F}_2^\dagger + h^3\mathcal{F}_3^\dagger + \cdots\right)\right)_s = 0. \tag{32}$$

Let us collect linear terms in h in (32):

$$\langle \underline{p}|\mathcal{F}_1|\underline{k}\rangle = \frac{\langle \underline{p}|\underline{V}|\underline{k}\rangle}{\omega(p) - \omega(k)} \qquad \text{for} \quad n(p) \neq n(k). \tag{33}$$

From here on, we use compact notation $|\underline{k}\rangle = \underline{|k_1, \ldots, k_{n(k)}\rangle}$, $\langle \underline{p}| = \underline{\langle p_{n(p)}, \ldots, p_1|}$, $\omega(p) = \omega(p_1) + \cdots + \omega(p_{n(p)})$ and so on. Equation (33) defines $(\mathcal{F}_1)_s$, but does not impose restrictions on $(\mathcal{F}_1)_d$. We fix the latter by the condition $(\mathcal{F}_1)_d = 0$.

In the second order in h, one finds from (32):

$$\left(\frac{\mathcal{F}_1^2}{2}\underline{\mathcal{H}}_0 + \underline{\mathcal{H}}_0\frac{\mathcal{F}_1^2}{2} - \mathcal{F}_1\underline{\mathcal{H}}_0\mathcal{F}_1 + [\Lambda, \underline{\mathcal{H}}_0] + [\mathcal{F}_1, \underline{V}]\right)_s = 0. \tag{34}$$

This equation defines Λ_s. We put $\Lambda_d = 0$ and insert the intermediate state decomposition

$$1 = \sum_q |\underline{q}\rangle\langle\underline{q}| \equiv |\underline{0}\rangle\langle\underline{0}| + \sum_{n(q)=1}^{\infty} \frac{1}{n(q)!} \int_{-\infty}^{\infty} |\underline{q}_{n(q)}, \ldots, \underline{q}_1\rangle\langle\underline{q}_1, \ldots, \underline{q}_{n(q)}| \prod_{j=1}^{n(q)} \frac{\mathrm{d}q_j}{2\pi}$$

into (34), providing

$$\langle \underline{p}|\Lambda|\underline{k}\rangle = \frac{1}{\omega(k)-\omega(p)}\left\{\sum_{\substack{q\\ n(p)\neq n(q)\neq n(k)}} \frac{\langle \underline{p}|\underline{V}|\underline{q}\rangle\langle \underline{q}|\underline{V}|\underline{k}\rangle}{[\omega(q)-\omega(p)][\omega(q)-\omega(k)]}\left[\omega(q)-\frac{\omega(p)+\omega(k)}{2}\right]\right.$$
$$\left. + \sum_{\substack{q\\ n(q)=n(k)}} \frac{\langle \underline{p}|\underline{V}|\underline{q}\rangle\langle \underline{q}|\underline{V}|\underline{k}\rangle}{[\omega(q)-\omega(p)]} + \sum_{\substack{q\\ n(q)=n(p)}} \frac{\langle \underline{p}|\underline{V}|\underline{q}\rangle\langle \underline{q}|\underline{V}|\underline{k}\rangle}{[\omega(q)-\omega(k)]}\right\}$$
$$\text{for}\quad n(p)\neq n(k). \tag{35}$$

Note that one can drop all underlines on the right-hand sides of equations (33) and (35), since $\langle \underline{\Phi}'|\underline{A}|\underline{\Phi}\rangle = \langle \Phi'|A|\Phi\rangle$. Similarly, we put $Y_d = 0$, since equation (32) (in the third order in h) determines Y_s only.

In the rest of this section we shall consider how the Hamiltonian $\mathcal{H}$ acts in the subspaces with zero, one and two renormalized fermions.

5.1. Vacuum sector

In the vacuum sector, one obtains from (33), (34) the standard Rayleigh–Schrödinger expansion (29) for the IFT ground-state energy:

$$E_{\text{vac}} = \langle \underline{0}|\mathcal{H}|\underline{0}\rangle = \langle \underline{0}|U(h)(\underline{\mathcal{H}}_0 + h\underline{V})U(h)^{-1}|\underline{0}\rangle = h\langle 0|V|0\rangle + \delta_2 E_{\text{vac}} + \delta_3 E_{\text{vac}} + O(h^4),$$

where

$$\delta_2 E_{\text{vac}} = -h^2 \sum_{\substack{q\\ n(q)\neq 0}} \frac{\langle 0|V|q\rangle\langle q|V|0\rangle}{\omega(q)},$$
$$\delta_3 E_{\text{vac}} = +h^3\left\{ -\langle 0|V|0\rangle \sum_{\substack{q\\ n(q)\neq 0}} \frac{\langle 0|V|q\rangle\langle q|V|0\rangle}{[\omega(q)]^2} + \sum_{\substack{q,q'\\ n(q)\neq 0\neq n(q')}} \frac{\langle 0|V|q\rangle\langle q|V|q'\rangle\langle q'|V|0\rangle}{\omega(q)\omega(q')}\right\}. \tag{36}$$

5.2. One-fermion sector

In the one-fermion sector $n(p) = n(k) = 1$, and we find

$$\langle \underline{p}|\mathcal{H}|\underline{k}\rangle = 2\pi\delta(p-k)\omega(p) + h\langle p|V|k\rangle + \delta_2\langle \underline{p}|\mathcal{H}|\underline{k}\rangle + \delta_3\langle \underline{p}|\mathcal{H}|\underline{k}\rangle + O(h^4), \tag{37}$$

where

$$\delta_2\langle \underline{p}|\mathcal{H}|\underline{k}\rangle = -\frac{h^2}{2}\sum_{\substack{q\\ n(q)\neq n(p)}} \langle p|V|q\rangle\langle q|V|k\rangle\left[\frac{1}{\omega(q)-\omega(p)} + \frac{1}{\omega(q)-\omega(k)}\right], \tag{38}$$

$$\delta_3\langle \underline{p}|\mathcal{H}|\underline{k}\rangle = +\frac{h^3}{2}\sum_{q,q'}\langle p|V|q\rangle\langle q|V|q'\rangle\langle q'|V|k\rangle\left\{[1-\delta_{n(q),n(p)}][1-\delta_{n(q'),n(p)}]\right.$$
$$\cdot\left[\frac{1}{[\omega(p)-\omega(q)]}\frac{1}{[\omega(p)-\omega(q')]} + \frac{1}{[\omega(k)-\omega(q)]}\frac{1}{[\omega(k)-\omega(q')]}\right]$$
$$\left. + \frac{1}{\omega(q)-\omega(q')}\left[\frac{\delta_{n(q),n(p)}[1-\delta_{n(q'),n(p)}]}{\omega(q')-\omega(p)} - \frac{[1-\delta_{n(q),n(p)}]\delta_{n(q'),n(p)}}{\omega(q)-\omega(k)}\right]\right\}. \tag{39}$$

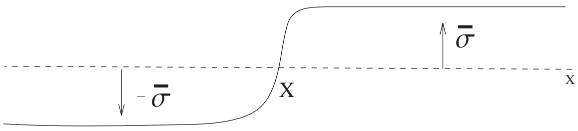


Figure 2. One-fermion state (42) represents a kink centred at X.

First, let us consider the linear term in h on the right-hand side of (37):

$$h\langle p|V|k\rangle = -h\int_{-\infty}^{\infty} \mathrm{dx}\langle p|\sigma(\mathrm{x})|k\rangle, \tag{40}$$

where

$$\langle p|\sigma(\mathrm{x})|k\rangle = \frac{\mathrm{i}\bar{\sigma}\exp[\mathrm{ix}(k-p)]}{p-k}\,\frac{\omega(p)+\omega(k)}{[\omega(p)\omega(k)]^{1/2}}$$

is the formfactor of the order spin operator (4) in the momentum basis. The integration in x in (40) leads to the divergent result

$$h\langle p|V|k\rangle = -2\pi\mathrm{i}\delta(p-k)\frac{f_0}{p-k}. \tag{41}$$

This singularity is well known in the standard formfactor perturbation theory, where it appears as the divergence of the first-order correction to the fermion mass, which is interpreted as a formal indication of confinement [1, 8].

To give a meaning to equation (41), let us mention that the generalized function $\delta(q)/q$ is well defined and equivalent to $-\delta'(q)$ in the class of the main functions $\varphi(q) \in C^1$ taking a zero value at the origin, $\varphi(0) = 0$. So, one can formally write

$$\delta(p-k)\frac{f_0}{p-k} = -f_0\delta'(p-k) + C\delta(p-k)$$

with some indeterminate constant C.

To get further insight, it is instructive to consider the matrix element $\langle \mathrm{X}|hV|k\rangle$, where the state $\langle \mathrm{X}|$ describes a 'bare' quark located at the point X:

$$\langle \mathrm{X}| = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{\mathrm{i}p\mathrm{X}}\langle p|. \tag{42}$$

For the matrix element of the order spin operator $\sigma(\mathrm{x})$, we get

$$\begin{aligned}\langle \mathrm{X}|\sigma(\mathrm{x})|k\rangle &= \fint_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{i}\bar{\sigma}\,\frac{\exp[\mathrm{i}p(\mathrm{X}-\mathrm{x})+\mathrm{i}k\mathrm{x}]}{p-k}\,\frac{\omega(p)+\omega(k)}{\sqrt{\omega(p)\omega(k)}}\\ &= \mathrm{sign}(\mathrm{x}-\mathrm{X})\bar{\sigma}\,\mathrm{e}^{\mathrm{i}k\mathrm{X}} + \mathrm{i}\bar{\sigma}\,\mathrm{e}^{\mathrm{i}k\mathrm{x}}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\,\frac{\exp[\mathrm{i}p(\mathrm{X}-\mathrm{x})]}{p-k}\\ &\quad\times\left\{\left[\frac{\omega(p)}{\omega(k)}\right]^{1/4} - \left[\frac{\omega(k)}{\omega(p)}\right]^{1/4}\right\}^2.\end{aligned} \tag{43}$$

Here the first term on the right-hand side is non-local, while the second term is well localized near the diagonal $\mathrm{x} = \mathrm{X}$ exponentially vanishing for $|\mathrm{x}-\mathrm{X}| \gg m^{-1}$. Equation (43) allows one to interpret the one-fermionic state $\langle \mathrm{X}|$ as a kink of width $\sim m^{-1}$ centred at X, which divides the regions with magnetizations $-\bar{\sigma}$ to the left and $+\bar{\sigma}$ to the right sides of it; see figure 2.

Substitution of (43) into (40) yields after integration in x

$$h\langle \mathrm{X}|V|k\rangle = (f_0\mathrm{X} + C)\,\mathrm{e}^{\mathrm{i}k\mathrm{X}},$$

where $f_0 = 2h\bar{\sigma}$ is the 'bare string tension' and the constant C is proportional to the length of the system L being infinite in the thermodynamic limit. Thus, Hamiltonian (3) acts in the one-particle subspace of 'bare' quarks $\mathbb{F}_1$ as

$$\mathcal{P}_1\mathcal{H}\mathcal{P}_1 = \omega(\hat{p}) + f_0\hat{\mathrm{x}} + C,$$

where $\hat{\mathrm{x}}$ and $\hat{p}$ are the one-particle coordinate and momentum operators respectively. The same formula written for 'dressed' quarks

$$\underline{\mathcal{P}}_1\mathcal{H}\underline{\mathcal{P}}_1 = \varepsilon(\hat{p}) + f\hat{\mathrm{x}} + C_R \tag{44}$$

gives us the perturbative definition of the renormalized quark dispersion law $\varepsilon(p)$ and renormalized string tension f. In the momentum representation, (44) takes the form

$$\langle \underline{p}|\mathcal{H}|\underline{k}\rangle = 2\pi\delta(p-k)[\varepsilon(p) + C_R] + 2\pi\mathrm{i}f\delta'(p-k),$$

which should be compared with (37) order by order in h.

The second-order term in (37) determines the leading correction to the quark energy $\delta_2\varepsilon(p)$ in expansion (24b):

$$\delta_2\langle \underline{p}|\mathcal{H}|\underline{k}\rangle = 2\pi\delta(p-k)[\delta_2\varepsilon(p) + \delta_2 E_{\mathrm{vac}}],$$

where $\delta_2 E_{\mathrm{vac}}$ is given by (36). Explicitly, it can be described either by the formfactor expansion following from (38)

$$\delta_2\varepsilon(p) = \delta_{2,3}\varepsilon(p) + \delta_{2,5}\varepsilon(p) + \cdots, \tag{45}$$

$$\delta_{2,n}\varepsilon(p) = -\frac{h^2}{n!}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1\cdots\mathrm{d}q_n}{(2\pi)^{n-1}}\,\frac{\delta(q_1+\cdots+q_n-p)}{\omega(q_1)+\ldots+\omega(q_n)-\omega(p)} \cdot \lim_{k\to p}\langle p|\sigma(0)|q_1,\ldots,q_n\rangle\langle q_n,\ldots,q_1|\sigma(0)|k\rangle, \tag{46}$$

or by the equivalent integral representation

$$\delta_2\varepsilon(p) = -h^2\int_{-\infty}^{\infty}\mathrm{dx}\int_0^{\infty}\mathrm{dy}\,\lim_{k\to p}\langle p|\sigma(\mathrm{x},\mathrm{y})(1-\mathcal{P}_1)\sigma(0,0)|k\rangle, \tag{47}$$

where $\sigma(\mathrm{x},\mathrm{y}) = \exp(-\mathrm{i}\mathrm{x}\hat{P} + \mathrm{y}\mathcal{H}_0)\sigma(0)\exp(\mathrm{i}\mathrm{x}\hat{P} - \mathrm{y}\mathcal{H}_0)$.

Representations (45)–(47) were first obtained and studied by Fonseca and Zamolodchikov [2, 7, 23]; we quoted their results in section 4 (see equations (25), (26), (28)). We determine the exact large-p asymptotics of the integral (46) for $n = 3$:

$$\delta_{2,3}\varepsilon(p) = \frac{\lambda^2 m^2}{2p}\left(\frac{1}{16} + \frac{1}{12\pi^2}\right) + O(p^{-2}), \tag{48}$$

which leads to (27).

The third-order term (39) in (37) contributes to both the string tension f and the quark energy $\varepsilon(p)$. It determines $\delta_3\varepsilon(p)$ and the constant a_3 in expansion (24a) for the quark mass m_R. Calculation of a_3 would be of much interest for interpreting the recent numerical calculation of the masses of the lightest mesons; see figure 7 in [2].

5.3. Two-fermion sector

In the two-particle sector of 'dressed' quarks, the Hamiltonian acts as

$$\langle \underline{p_2, p_1}|\mathcal{H}|\underline{k_1, k_2}\rangle = (2\pi)^2[\omega(p_1)+\omega(p_2)][\delta(p_1-k_1)\delta(p_2-k_2)-\delta(p_1-k_2)\delta(p_2-k_1)] \\ + h\langle p|V|k\rangle + \delta_2\langle \underline{p}|\mathcal{H}|\underline{k}\rangle + \delta_3\langle \underline{p}|\mathcal{H}|\underline{k}\rangle, +O(h^4), \tag{49}$$

where $\delta_2\langle \underline{p}|\mathcal{H}|\underline{k}\rangle$ and $\delta_3\langle \underline{p}|\mathcal{H}|\underline{k}\rangle$ are given by equations (38) and (39), respectively, with $n(p) = n(k) = 2$. Two initial terms on the right-hand side of (49) give rise to the 'bare' Bethe–Salpeter equation (9).

The explicit form of the second-order correction is

$$\delta_2\langle \underline{p_2, p_1}|\mathcal{H}|\underline{k_1, k_2}\rangle = -4\pi f_0\delta(p_1+p_2-k_1-k_2)\delta_2\mathcal{G}(p_1, p_2|k_1, k_2),$$

where

$$\delta_2\mathcal{G}(p_1, p_2|k_1, k_2) = \frac{h}{8\bar{\sigma}}\sum_{j=2}^{\infty}\frac{1}{(2j)!}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_{2j}}{(2\pi)^{2j-1}}\delta(p_1+p_2-q_1-\ldots-q_{2j}) \\ \cdot\langle p_2, p_1|\sigma(0)|q_1,\ldots,q_{2j}\rangle\langle q_{2j},\ldots,q_1|\sigma(0)|k_1,k_2\rangle \\ \cdot\left[\frac{1}{\omega(q_1)+\cdots+\omega(q_{2j})-\omega(p_1)-\omega(p_2)}\right. \\ \left.+\frac{1}{\omega(q_1)+\cdots+\omega(q_{2j})-\omega(k_1)-\omega(k_2)}\right]. \tag{50}$$

Application of the Wick expansion to the formfactors in the integrand breaks (50) into a sum of diagrams. Some of them contain one or two products of the form

$$\langle p_\alpha|\sigma(0)|q_\gamma\rangle\langle q_\gamma|\sigma(0)|k_\beta\rangle = \frac{\omega(p_\alpha)+\omega(q_\gamma)}{[\omega(p_\alpha)\omega(q_\gamma)]^{1/2}}\frac{\omega(k_\beta)+\omega(q_\gamma)}{[\omega(k_\beta)\omega(q_\gamma)]^{1/2}}\mathcal{P}\frac{1}{p_\alpha-q_\gamma}\cdot\mathcal{P}\frac{1}{k_\beta-q_\gamma}, \tag{51}$$

which have two kinematic singularities in the integration variable q_γ at $q_\gamma = p_\alpha$ and $q_\gamma = k_\beta$. Here, $\mathcal{P}\frac{1}{z}$ denotes the 'principal value' generalized function:

$$\mathcal{P}\frac{1}{z} = \frac{1}{2}\left(\frac{1}{z+\mathrm{i}0}+\frac{1}{z-\mathrm{i}0}\right).$$

Let us rewrite the singular factor on the right-hand side of (51) as

$$\mathcal{P}\frac{1}{p_\alpha-q_\gamma}\cdot\mathcal{P}\frac{1}{k_\beta-q_\gamma} = \mathcal{P}\frac{1}{(p_\alpha-q_\gamma)(k_\beta-q_\gamma)}+\pi^2\delta(p_\alpha-k_\beta)\delta(p_\alpha-q_\gamma), \tag{52}$$

where

$$\mathcal{P}\frac{1}{(p_\alpha-q_\gamma)(k_\beta-q_\gamma)} = \frac{1}{2}\left[\frac{1}{(p_\alpha-q_\gamma-\mathrm{i}0)(k_\beta-q_\gamma-\mathrm{i}0)}+\frac{1}{(p_\alpha-q_\gamma+\mathrm{i}0)(k_\beta-q_\gamma+\mathrm{i}0)}\right].$$

Substitution of (52) into factor (51) leads to splitting of diagrams containing (one or two) such factors into several (two or four) terms. The resulting diagrams can be classified by the number of δ-functions $\delta(p_\alpha-k_\beta)$ arising from the second term on the right-hand side of (52).

(i) Diagrams with two δ-functions give rise to the vacuum energy correction $\delta_2 E_{\text{vac}}$.

(ii) Diagrams with one δ-function contribute to the corrections $\delta_2\varepsilon(p_1)$, $\delta_2\varepsilon(p_2)$ to the energies of two quarks.

(iii) The rest of the diagrams are regular at $p_\alpha = k_\beta$ for $\alpha, \beta = 1, 2$. We denote their sum by $\delta_2\mathcal{G}^{(\text{reg})}(p_1, p_2|k_1, k_2)$. It determines (to the linear order in h) the kernel $\Delta\mathbb{G}_P^{(\text{reg})}(p|q)$ in the renormalized Bethe–Salpeter equation (22):

$$\Delta_1\mathbb{G}_P^{(\text{reg})}(p|k) = \delta_2\mathcal{G}^{(\text{reg})}(P/2+p, P/2-p|P/2+k, P/2-k).$$

6. Local multi-quark corrections to the meson masses

It is not difficult to account for perturbatively the regular correction term $\Delta_1\mathbb{G}_P^{(\mathrm{reg})}(p|k)$ in the Bethe–Salpeter equation (22) in both the low-energy and semiclassical expansions. The resulting local multi-quark correction to the meson energy reads as

$$\delta_3 E_n(P) = -\frac{f_0^2\omega^3(P/2)}{4m^2}\frac{\partial^2}{\partial p^2}\bigg|_{p=0}\Delta_1\mathbb{G}_P^{(\mathrm{reg})}(p|p) \tag{53}$$

in the low-energy case $n \sim 1$, and

$$\delta_3 E_n(P) = -\frac{2f_0^2}{|(p_1-p_2)v|}\Delta_1\mathbb{G}_P^{(\mathrm{reg})}\left(\frac{p_1-p_2}{2}\bigg|\frac{p_1-p_2}{2}\right) \tag{54}$$

in the semiclassical case $n \gg 1$. Here, momenta p_1 and p_2 are the solutions of equations

$$p_1 + p_2 = P, \qquad \omega(p_1)+\omega(p_2) = \left(P^2+M_n^2\right)^{1/2},$$

and

$$v = \frac{p_1}{\omega(p_1)} - \frac{p_2}{\omega(p_2)}.$$

To obtain the local multi-quark corrections to the meson masses, we rewrite (53) and (54) in the rapidity variables $\beta_1 = \operatorname{arcsinh}(p_1/m)$ and $\beta_2 = \operatorname{arcsinh}(p_2/m)$, respectively, and then proceed to the limit $P \to \infty$, yielding

$$\frac{\delta_3 M_n^2}{m^2} = -\frac{\lambda^3}{8}\lim_{\beta_1\to\infty}\left(\frac{\partial}{\partial\beta_1}-\frac{\partial}{\partial\beta_2}\right)^2\bigg|_{\beta_1=\beta_2}\frac{m^2 W(\beta_1,\beta_2)}{\bar\sigma^2} \tag{55}$$

in the low-energy case and

$$\frac{\delta_3 M_n^2}{m^2} = -\frac{\lambda^3 m^2}{M_n^2-4m^2}\lim_{\beta\to\infty}\frac{m^2 W(\beta+\eta,\beta-\eta)}{\bar\sigma^2} \tag{56}$$

in the semiclassical case. Here $\eta = \operatorname{arccosh}[M_n/(2m)]$, and

$$W(\beta_1,\beta_2) = \frac{4\bar\sigma}{h}\omega(p_1)\omega(p_2)\Delta_1\mathbb{G}_P^{(\mathrm{reg})}\left(\frac{p_1-p_2}{2}\bigg|\frac{p_1-p_2}{2}\right), \tag{57}$$

where $p_j = m\sinh\beta_j$ for $j = 1, 2$. Function $W(\beta_1,\beta_2)$ determined by (57) admits a compact integral representation, analogous to (47):

$$W(\beta_1,\beta_2) = \int_\infty^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y \lim_{\substack{\beta_1'\to\beta_1\\ \beta_2'\to\beta_2}}\langle\beta_2,\beta_1|\sigma(x,y)(1-\mathcal{P}_0-\mathcal{P}_2)\sigma(0,0)|\beta_1',\beta_2'\rangle. \tag{58}$$

In appendix B, we extract from this function the 'irreducible' part $W_{\mathrm{irr}}(\beta_1,\beta_2)$ which determines $\delta_3 M_n^2$. It is expressed there in terms of the two-fermion matrix elements of the order spin operator pairs, which are explicitly known [23].

The third-order term $\delta_3\langle \underline{p}|\mathcal{H}|\underline{k}\rangle$ in (49) also contains regular and singular parts. The former contributes to the meson masses starting from the fourth order. The latter renormalizes the quark dispersion law and the string tension, which give rise to the 'nonlocal' third-order multi-quark correction to the meson energy $E_n(P)$. It is expected [2], however, that *in the limit* $P \to \infty$ the nonlocal multi-quark corrections to $E_n(P)$ can be absorbed by renormalization of the meson mass and string tension. Thus, the third-order multi-quark correction to the meson masses should be described by representations (55) and (56), in which the 'bare' parameters should be replaced by the renormalized ones $m \to m_q$, $\lambda \to \lambda_R$, where $\lambda_R = f/m_q^2$, and m_q and f are given by expansions (24*a*) and (24*c*), respectively.

7. Conclusions

This paper is devoted to an extension to the third order of the weak-h expansions for the meson masses $M_n(h)$ in the ferromagnetic IFT. There are four third-order contributions to it. The first one comes from the Bethe–Salpeter equation (13) written in the two-quark approximation. For the semiclassical expansion this contribution is described by equations (16)–(19), for the low-energy expansion was already determined in [2] (see (15)). Three other contributions to $M_n(h)$ are due to the multi-fermion fluctuations. The local contribution arises from the regular radiative correction to the Bethe–Salpeter kernel. For this contribution, we obtain the integral representations (55)–(58), which are compact and appropriate for analytical analysis, and representations (B.14)–(B.16), which we plan to use in the future numerical calculations. The last two multi-quark contributions to the meson mass come from the third-order corrections $\delta_3\varepsilon(p)$ and $\delta_3 f$ to the quark self-energy and string tension, which are contained implicitly in the formfactor expansion (39). Explicit extraction of $\delta_3\varepsilon(p)$ and $\delta_3 f$ from (39) is rather involved. Whereas for $\delta_3 f$ the result is essentially known (see (24c), (29), (30)), an explicit calculation of the third-order correction to the quark self-energy and quark mass remains an open problem.

Acknowledgments

I am grateful to A B Zamolodchikov for stimulating and interesting discussions. This work is supported by the Belarusian Republican Foundation for Fundamental Research under grant Φ 07-147.

Appendix A. Perturbative solution of the Bethe–Salpeter equation

In this appendix, we describe a perturbative solution of the 'bare' Bethe–Salpeter equation (13) in the infinite momentum frame $P \to \infty$ to the third order in the magnetic field h.

A.1. Some exact relations

It is convenient to rewrite equation (13) in new notation

$$\phi(u) = \Phi(u)(1-u^2)^{-1/2}, \qquad \nu^2 = \frac{\widetilde{M}^2 - 4m^2}{\widetilde{M}^2}, \qquad \rho = \frac{2h\bar{\sigma}}{\widetilde{M}^2}.$$

Since $\phi(-u) = -\phi(u)$, equation (13) takes the form

$$(u^2-\nu^2)\phi(u) = \rho \fint_{-1}^{1} \frac{\mathrm{d}v}{\pi}\phi(v)\left[\frac{uv}{2} + 4\frac{1-uv}{(u-v)^2}\right], \tag{A.1}$$

or equivalently

$$(u^2-\nu^2)\phi(u) = 2\mathrm{i}a\rho u + 4\rho[-u + (1-u^2)\partial_u] \fint_{-1}^{1} \frac{\mathrm{d}v}{\pi}\frac{\phi(v)}{v-u}, \tag{A.2}$$

where

$$a = \int_{-1}^{1} \frac{\mathrm{d}v}{4\pi\mathrm{i}} v\phi(v).$$

We shall require that $\phi(u)$ is a purely imaginary function in the interval $(-1, 1)$.

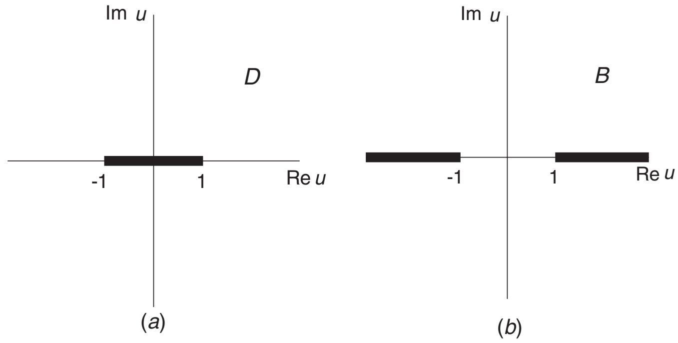


Figure A1. (*a*) D is the region of analyticity of $g(u)$ and (*b*) B is the region of analyticity of $\mathbf{U}(u)$,

Set

$$g(u) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\mathrm{d}v}{v-u} \phi(v), \tag{A.3}$$

for complex $u \notin [-1, 1]$. The function $g(u)$ is analytical in the region D shown in figure A1(*a*), and

$$g(u) = -\frac{2a}{u^2} + O(u^{-3}) \qquad \text{at} \quad u \to \infty,$$

providing

$$2a = \mathrm{Res}|_{u=\infty}[g(u)u]. \tag{A.4}$$

For real $u \in (-1, 1)$, we get

$$\phi(u) = g(u + \mathrm{i}0) - g(u - \mathrm{i}0), \qquad \fint_{-1}^{1} \frac{\mathrm{d}v}{\pi} \frac{\phi(v)}{v-u} = \mathrm{i}[g(u + \mathrm{i}0) + g(u - \mathrm{i}0)],$$

and equation (A.2) takes the form

$$(u^2 - \nu^2)[g(u + \mathrm{i}0) - g(u - \mathrm{i}0)] = 4\mathrm{i}\rho[-u + (1 - u^2)\partial_u][g(u + \mathrm{i}0) + g(u - \mathrm{i}0)] + 2\mathrm{i}a\rho u. \tag{A.5}$$

Let us define two functions in D:

$$\begin{aligned} \mathbf{U}_1(u) &= -4\mathrm{i}\rho[-u + (1 - u^2)\partial_u]g(u) + (u^2 - \nu^2)g(u) - \mathrm{i}a\rho u, \\ \mathbf{U}_2(u) &= 4\mathrm{i}\rho[-u + (1 - u^2)\partial_u]g(u) + (u^2 - \nu^2)g(u) + \mathrm{i}a\rho u. \end{aligned} \tag{A.6}$$

Due to (A.5), we have $\mathbf{U}_1(u + \mathrm{i}0) = \mathbf{U}_2(u - \mathrm{i}0)$ for $u \in (-1, 1)$. Therefore, function $\mathbf{U}(u)$ defined as

$$\mathbf{U}(u) = \begin{cases} \mathbf{U}_1(u) & \text{for } \mathrm{Im}\, u > 0 \\ \mathbf{U}_2(u) & \text{for } \mathrm{Im}\, u < 0 \end{cases}$$

can be analytically continued into the complex region B shown in figure A1(*b*). Note that $\mathbf{U}(u)$ is even in B and real in the interval $(-1, 1)$.

Let us solve the differential equation (A.6) with respect to $g(u)$:

$$g(u) = \frac{\mathrm{i}}{4\rho} \int_{-\mathrm{i}\infty}^{u} \mathrm{d}v \frac{\mathbf{U}_{+}(v) \exp\left[\frac{\mathrm{i}}{4\rho}(u - v)\right]}{[(1 - u^2)(1 - v^2)]^{1/2}} \left[\frac{(1 - u)(1 + v)}{(1 + u)(1 - v)}\right]^{\mathrm{i}(1-\nu^2)/(8\rho)}, \tag{A.7}$$

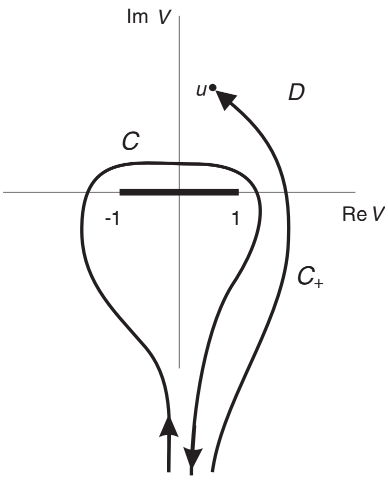


Figure A2. Integration paths in equations (A.7) and (A.8).

where $\mathbf{U}_+(u) = \mathbf{U}_1(u) + \mathrm{i}a\rho u$, and the branch of the last factor in the integrand is fixed as

$$\arg\left[\frac{1-(u+\mathrm{i}0)}{1+(u+\mathrm{i}0)}\right] = \arg\left[\frac{1+(v+\mathrm{i}0)}{(1-(v+\mathrm{i}0)}\right] = 0 \qquad \text{for real} \quad u, v \in (-1, 1).$$

The integration in (A.7) goes along the path C_+, as shown in figure A2.

Function $g(u)$ should be single valued in D. The trivial monodromy behaviour of $g(u)$ at $u = \infty$ is provided by (A.7), if the following requirement is satisfied:

$$\int_C \mathrm{d}v \frac{\mathbf{U}_+(v)}{(1-v^2)^{1/2}} \left[\frac{(1+v)}{(1-v)}\right]^{\mathrm{i}(1-\nu^2)/(8\rho)} \exp\left[-\frac{\mathrm{i}v}{4\rho}\right] = 0, \tag{A.8}$$

where the integration path C is shown in figure A2. The last condition determines the spectrum ν_n. If $\rho \ll 1$, the integral in (A.8) is determined by the contributions of the saddle points $v = \pm\nu$ of the function $\Upsilon(v)$:

$$\Upsilon(v) = \frac{1-\nu^2}{2} \ln\left(\frac{1+v}{1-v}\right) - v. \tag{A.9}$$

In the semiclassical limit $n \gg 1$, the contributions of these two saddle points are well separated, and (A.8) yields the final asymptotical formula

$$-\frac{\Upsilon(\nu_n)}{4} = \rho\pi\left(n - \frac{1}{4}\right) + \rho \arg\left\langle \frac{\mathbf{U}_+(\nu_n + \Delta v)}{\sqrt{1-(\nu_n+\Delta v)^2}} \exp\left[\frac{\mathrm{i}}{4\rho}\Delta\Upsilon(\nu_n + \Delta v)\right]\right\rangle, \tag{A.10}$$

valid to all orders in $\rho \to 0$. Here

$$\Delta\Upsilon(\nu + \Delta v) = \Upsilon(\nu + \Delta v) - \Upsilon(\nu) - \frac{\nu(\Delta v)^2}{1-\nu^2};$$

averaging $\langle \ldots \rangle$ stands for

$$\langle f(\Delta v)\rangle = \int_{-\infty}^{\infty} \mathrm{d}\Delta v f(\Delta v) \exp\left[\frac{\mathrm{i}\nu(\Delta v)^2}{4\rho(1-\nu^2)}\right] \left\{\int_{-\infty}^{\infty} \mathrm{d}\Delta v' \exp\left[\frac{\mathrm{i}\nu(\Delta v')^2}{4\rho(1-\nu^2)}\right]\right\}^{-1},$$

providing

$$\langle(\Delta v)^{2j+1}\rangle = 0, \qquad \langle(\Delta v)^{2j}\rangle = [4\mathrm{i}\rho(1-\nu^2)/\nu]^j \Gamma(1/2+j)/\Gamma(1/2). \tag{A.11}$$

At small ρ, Δv, function $\mathbf{U}_+(\nu + \Delta v)$ can be expanded as

$$\mathbf{U}_+(\nu + \Delta v) = 1 + \sum_{i=1}^{\infty} \sum_{l=0}^{\infty} c_{il} \rho^i (\Delta v)^l \tag{A.12}$$

under appropriate normalization of $\phi(u)$.

A.2. Perturbation expansion

To obtain the explicit semiclassical expansion for the spectrum ν_n, we need

(i) to calculate several initial terms in expansion (A.12),
(ii) to substitute (A.12) into (A.10) and to expand the expression in $\langle \ldots \rangle$ in powers of Δv,
(iii) to perform averaging in (A.10) by use of (A.11),
(iv) to expand $\arg\langle \ldots \rangle$ in (A.10) in powers of ρ.

Steps (ii)–(iv) are straightforward and well suitable for computer calculations; below, we describe only step (i).

Let us write down the formal Neumann series solving equation (A.1) in the class of the generalized functions:

$$\phi(u) = \phi^{(0)}(u) + \phi^{(1)}(u) + O(\rho^2), \tag{A.13}$$

$$\begin{aligned} \phi^{(0)}(u) &= \frac{\pi}{\mathrm{i}\nu}[\delta(u-\nu) - \delta(u+\nu)], \\ \phi^{(1)}(u) &= \frac{\rho u}{\mathrm{i}} \left[\frac{1}{u^2-\nu^2} - 8\frac{1}{(u^2-\nu^2)^2} + 16\frac{1-\nu^2}{(u^2-\nu^2)^3} \right]; \end{aligned} \tag{A.14}$$

the principal values are implied for the singular terms in (A.14). Substitution of (10) into (A.3), (A.4), (A.6) yields

$$\begin{aligned} g(u) &= g^{(0)}(u) + g^{(1)}(u) + O(\rho^2), \\ a &= a^{(0)} + a^{(1)} + O(\rho^2), \\ \mathbf{U}(u) &= \mathbf{U}^{(0)}(u) + \mathbf{U}^{(1)}(u) + O(\rho^2), \end{aligned}$$

where

$$\begin{aligned} g^{(0)}(u) &= \frac{1}{u^2-\nu^2}, \qquad a^{(0)} = -\frac{1}{2}, \qquad \mathbf{U}^{(0)}(u) = 1, \\ a^{(1)} &= -\frac{\rho}{2\pi\nu^2}\left[-2 + \nu^2 + \frac{-2-2\nu^2+\nu^4}{2\nu} \ln\left(\frac{1-\nu}{1+\nu}\right)\right]. \end{aligned}$$

We skip lengthy expressions for $g^{(1)}(u)$ and $\mathbf{U}^{(1)}(u)$. Note that all singular terms at $u = \nu$ and at $u = -\nu$ cancel in $\mathbf{U}^{(1)}(u)$. The explicit expressions for the Taylor coefficients c_{il} in (A.12) read as

$$\begin{aligned} c_{10} &= -\frac{\mathrm{i}\nu}{2}, \\ c_{11} &= \frac{-6+10\nu^2+3\nu^4-3\nu^6}{3\pi\nu^3(\nu^2-1)^2} - \frac{2+\nu^4}{2\pi\nu^4} \ln\left(\frac{1-\nu}{1+\nu}\right) - \frac{\mathrm{i}}{2}, \\ c_{12} &= \frac{-9+27\nu^2-37\nu^4+3\nu^6}{6\pi\nu^4(\nu^2-1)^3} - \frac{\nu^2-3}{4\pi\nu^5} \ln\left(\frac{1-\nu}{1+\nu}\right), \\ \operatorname{Im} c_{20} &= \nu a^{(1)}/\rho. \end{aligned}$$

These constants are sufficient to obtain the equation determining ν_n to the third order in ρ:

$$-\frac{\Upsilon(\nu_n)}{4} = \rho\pi\left(n-\frac{1}{4}\right)+\rho^2\frac{5-6\nu_n^2-9\nu_n^4+6\nu_n^6}{12\nu_n^3\left(1-\nu_n^2\right)} + \rho^3\left[\frac{30-62\nu_n^2+54\nu_n^4-21\nu_n^6+18\nu_n^8-3\nu_n^{10}}{6\pi\nu_n^5\left(\nu_n^2-1\right)^2} + \frac{10-4\nu_n^2+6\nu_n^4+4\nu_n^6-\nu_n^8}{4\pi\nu_n^6}\ln\left(\frac{1-\nu_n}{1+\nu_n}\right)\right]+O(\rho^4). \quad (A.15)$$

Returning in equation (A.15) to the variables θ_n and λ which are related to ν_n and ρ by

$$\nu_n = \tanh\theta_n, \qquad \rho = \frac{\lambda}{4\cosh^2\theta_n},$$

we come finally to expansion (16)–(19).

Appendix B. Integral of the four-particle matrix element

It was shown in section 6 that the local third-order multi-quark corrections to the meson mass can be expressed in terms of the integral of the four-particle matrix element:

$$W(\beta_1,\beta_2) = \int_{-\infty}^{\infty}\mathrm{dx}\int_0^{\infty}\mathrm{dy}\lim_{\substack{\beta_1'\to\beta_1\\ \beta_2'\to\beta_2}}\langle\beta_2',\beta_1'|\sigma(\mathrm{x},\mathrm{y})(1-\mathcal{P}_0-\mathcal{P}_2)\sigma(0,0)|\beta_1,\beta_2\rangle \quad (B.1)$$

over the Euclidean half-plane (see equations (55) and (56)). In this section, we extract from $W(\beta_1,\beta_2)$ the most important 'irreducible' part $W_{\mathrm{irr}}(\beta_1,\beta_2)$ and transform it to a form suitable for numerical calculations.

It is straightforward to rewrite formula (54), giving the local third-order semiclassical correction to the meson energy $\delta_3E_n(P)$, in terms of $W(\beta_1,\beta_2)$:

$$\delta_3E_n(P) = -\left(\frac{h\bar\sigma}{m^2}\right)^3\frac{4m^6}{E_n(P)\left(M_n^2-4m^2\right)}\frac{W(\beta_1,\beta_2)}{\bar\sigma^2}. \quad (B.2)$$

Here

$$P = m(\sinh\beta_1+\sinh\beta_2), \qquad E_n = m(\cosh\beta_1+\cosh\beta_2)$$

are the meson momentum and energy, and β_1 and β_2 are the rapidities of two quarks (forming the meson) at their collision; $M_n = \left(E_n^2-P^2\right)^{1/2}$ is the meson mass.

The matrix element in the integrand in (B.1) can be expanded by the use of the Wick rule [23]:

$$G(\beta_2,\beta_1|\beta_1,\beta_2) \equiv \lim_{\substack{\beta_1'\to\beta_1,\\ \beta_2'\to\beta_2}}\langle\beta_2',\beta_1'|\sigma(\mathrm{x},\mathrm{y})\sigma(0,0)|\beta_1,\beta_2\rangle = \frac{G(\beta_1|\beta_1)G(\beta_2|\beta_2)}{G} - \frac{G(\beta_1|\beta_2)G(\beta_2|\beta_1)}{G} - \frac{1}{G}\left(\frac{G(\beta_1,\beta_2)}{E(\beta_1)E(\beta_2)}\right)^2. \quad (B.3)$$

Here we follow the notation of [23]:

$$x = (\mathrm{x},\mathrm{y}) = (r\cos\theta, r\sin\theta), \qquad G(r) = \langle 0|\sigma(x)\sigma(0)|0\rangle,$$

$$G(r,\theta;\beta_1,\beta_2) = \langle 0|\sigma(x)\sigma(0)|\beta_1,\beta_2\rangle,$$

$$\langle\beta'|\sigma(x)\sigma(0)|\beta\rangle = 2\pi\delta(\beta'-\beta)+G(r,\theta;\beta'|\beta),$$

$$E(r,\theta;\beta) = \exp\left[\frac{\mathrm{i}mr}{2}\sinh(\beta+\mathrm{i}\theta)\right]$$

and drop the explicit indication of position dependence for the matrix elements. The equality

$$\langle \beta_2, \beta_1 | \sigma(x)\sigma(0)|0\rangle = -\frac{\langle 0|\sigma(x)\sigma(0)|\beta_1, \beta_2\rangle}{[E(r,\theta;\beta_1)E(r,\theta;\beta_2)]^2}$$

has been used in deriving (B.3). Explicit expressions for the matrix elements $G(r)$, $G(r,\theta;\beta_1,\beta_2)$, $G(r,\theta;\beta'|\beta)$ in terms of the solutions $\varphi(r)$, $\chi(r)$ of the sinh–Gordon equation and associated Lax functions $\Psi_+(r,\theta;\beta)$, $\Psi_-(r,\theta;\beta)$ are known from [23].

Proceeding to polar coordinates r,θ in integral (B.1), one can easily show that

$$\int_0^\pi \mathrm{d}\theta\, G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2) = \int_0^\pi \mathrm{d}\theta\, G(r,\theta;\beta_2+\beta,\beta_1+\beta|\beta_1+\beta,\beta_2+\beta) \qquad \text{(B.4)}$$

for arbitrary real β. The proof of (B.4) is based on the relations

$$G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2) = G(r,0;\beta_2+\mathrm{i}\theta,\beta_1+\mathrm{i}\theta|\beta_1+\mathrm{i}\theta,\beta_2+\mathrm{i}\theta), \qquad \text{(B.5)}$$

$$G(\beta_2+\mathrm{i}\pi,\beta_1+\mathrm{i}\pi|\beta_1+\mathrm{i}\pi,\beta_2+\mathrm{i}\pi) = G(\beta_2,\beta_1|\beta_1,\beta_2), \qquad \text{(B.6)}$$

which follow from (B.3) and similar properties of functions $G(\beta_1,\beta_2)$, $G(\beta_1|\beta_2)$; see [23]. Equality (B.4) means that the integral on its left-hand side is Lorentz invariant.

Unfortunately, the integral

$$\int_0^\infty r\,\mathrm{d}r \int_0^\pi \mathrm{d}\theta\, G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)$$

diverges at large r. It becomes convergent after subtraction of the 'reducible part' (see (B.1)):

$$G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2) \to G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2) - \lim_{\substack{\beta_1'\to\beta_1\\ \beta_2'\to\beta_2}} \langle \beta_2'\beta_1'|\sigma(r,\theta)(\mathcal{P}_0+\mathcal{P}_2)\sigma(0,0)|\beta_1,\beta_2\rangle. \qquad \text{(B.7)}$$

However, the second term on the right-hand side here does not satisfy the monodromy property such as (B.6). This means that the local multi-quark correction (B.2) to the meson energy is not Lorentz invariant by itself. We hope that the Lorentz invariance form of $\delta_3 E_n(P)$ will be restored in the third order in h after picking up all the terms contributing to it, as happens [2] for the second-order term $\delta_2 E_n(P)$.

At the moment, it is helpful to extract the 'Lorentz-invariant' term from the reducible part in (B.7). Namely, we shall subdivide it as follows:

$$\lim_{\substack{\beta_1'\to\beta_1\\ \beta_2'\to\beta_2}} \langle \beta_2',\beta_1'|\sigma(r,\theta)(\mathcal{P}_0+\mathcal{P}_2)\sigma(0,0)|\beta_1,\beta_2\rangle = \Delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2) + \delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2),$$

where the first term satisfies the required monodromy property

$$\Delta G(r,\theta;\beta_2+\mathrm{i}\pi,\beta_1+\mathrm{i}\pi|\beta_1+\mathrm{i}\pi,\beta_2+\mathrm{i}\pi) = \Delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2),$$

while the function $\delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)$ does not satisfy such a property, but the integral

$$\int_0^\infty r\,\mathrm{d}r \int_0^\pi \mathrm{d}\theta\, \delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)$$

converges at finite β_1,β_2 and vanishes in the infinite momentum limit $(\beta_1+\beta_2)/2 \to +\infty$.

Note that function $\Delta G(\beta_2,\beta_1|\beta_1,\beta_2)$ is analogous to function $S(\beta|\beta)$ defined by equation (5.13) on page 20 of [23], whereas $\delta G(\beta_2,\beta_1|\beta_1,\beta_2)$ is analogous to the zig-zag diagram (b) in figure 3 on page 19.

Let us obtain explicit expressions for $\Delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)$.

(i) Vacuum sector.

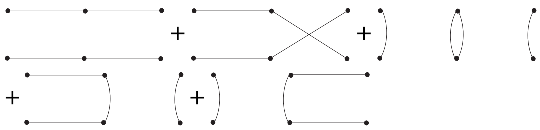
$$\langle \beta_2, \beta_1 | \sigma(r,\theta) \mathcal{P}_0 \sigma(0,0) | \beta_1, \beta_2 \rangle$$
$$= \bar{\sigma}^2 \tanh^2 \frac{\beta_1 - \beta_2}{2} \exp\{-\mathrm{i} r m [\sinh(\beta_1 + \mathrm{i}\theta) + \sinh(\beta_2 + \mathrm{i}\theta)]\}$$
$$= 2\bar{\sigma}^2 \tanh^2 \frac{\beta_1 - \beta_2}{2} \cos\{rm[\sinh(\beta_1 + \mathrm{i}\theta) + \sinh(\beta_2 + \mathrm{i}\theta)]\}$$
$$- \bar{\sigma}^2 \tanh^2 \frac{\beta_1 - \beta_2}{2} \exp\{\mathrm{i} r m [\sinh(\beta_1 + \mathrm{i}\theta) + \sinh(\beta_2 + \mathrm{i}\theta)]\}. \qquad \text{(B.8)}$$

The first and the second terms on the right-hand side of (B.8) should be assigned to $\Delta G(\beta_2, \beta_1 | \beta_1, \beta_2)$ and $\delta G(\beta_2, \beta_1 | \beta_1, \beta_2)$, respectively.

(ii) Two-quark sector.

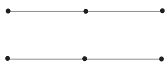
$$\langle \beta_2', \beta_1' | \sigma(r,\theta) \mathcal{P}_2 \sigma(0,0) | \beta_1, \beta_2 \rangle$$
$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta_1 \, \mathrm{d}\eta_2}{(2\pi)^2} \langle \beta_2', \beta_1' | \sigma(r,\theta) | \eta_2, \eta_1 \rangle \langle \eta_1, \eta_2 | \sigma(0,0) | \beta_1, \beta_2 \rangle.$$

This can be split into five diagrams:

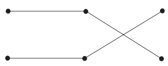


(B.9)

The contributions of the two former diagrams to $\Delta G(\beta_2, \beta_1 | \beta_1, \beta_2)$ are



$$\to \frac{1}{\bar{\sigma}^2} S(\beta_1|\beta_1) S(\beta_2|\beta_2), \qquad \text{(B.10)}$$



$$\to -\frac{1}{\bar{\sigma}^2} [R(\beta_1|\beta_2)]^2, \qquad \text{(B.11)}$$

where

$$R(\beta_1|\beta_2) = \bar{\sigma}^2 \exp[-\mathrm{i} m r (\sinh\beta_1 + \sinh\beta_2)/2] \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{2\pi} \exp[\mathrm{i} m r \sinh\eta] \cdot \coth\frac{\eta - \beta_1}{2}$$
$$\times \coth\frac{\eta - \beta_2}{2} + 2\bar{\sigma}^2 \coth\frac{\beta_1 - \beta_2}{2} \sin[mr(\sinh\beta_1 - \sinh\beta_2)/2]$$
$$+ \bar{\sigma}^2 \exp[\mathrm{i} m r (\sinh\beta_1 + \sinh\beta_2)/2] \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{2\pi} \exp[\mathrm{i} m r \sinh\eta']$$
$$\times \tanh\frac{\eta' - \beta_1}{2} \tanh\frac{\eta' - \beta_2}{2}, \qquad S(\beta_1|\beta_1) = \lim_{\beta_2 \to \beta_1} R(\beta_1|\beta_2). \qquad \text{(B.12)}$$

Note that $0 < \mathrm{Im}\beta_j < \pi$ for $j = 1, 2$ is supposed in (B.12). It is easy to verify that

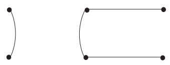
$$R(\beta_1 + \mathrm{i}\pi | \beta_2 + \mathrm{i}\pi) = R(\beta_1|\beta_2).$$

The third diagram in (B.9) is proportional to the function $f^{(2)}(t)$, which determines the well-known large-distance asymptotics of the Ising correlation function [24]. Its contribution to $\Delta G(\beta_2, \beta_1 | \beta_1, \beta_2)$ is

$$\to -2\bar{\sigma}^2 \cos\{rm[\sinh(\beta_1 + \mathrm{i}\theta) + \sinh(\beta_2 + \mathrm{i}\theta)]\} \tanh^2 \frac{\beta_1 - \beta_2}{2} f^{(2)}(mr),$$
$$f^{(2)}(t) = -\frac{1}{\pi^2} \left\{ \left[K_1^2(t) - K_0^2(t) \right] t^2 - t K_0(t) K_1(t) + \frac{1}{2} K_0^2(t) \right\},$$

where $K_0(t)$ and $K_1(t)$ are MacDonald's functions.

The fourth and the fifth diagrams are equal to one another and can be written as



$$= -\mathrm{i}\bar{\sigma}^2 \tanh\frac{\beta_1-\beta_2}{2}\{1+\exp[-\mathrm{i}mr[\sinh(\beta_1+i\theta)+\sinh(\beta_2+\mathrm{i}\theta)]]\}$$
$$\cdot\mathcal{T}_2(r,\beta_1+\mathrm{i}\theta,\beta_2+\mathrm{i}\theta)+\bar{\sigma}^2\tanh^2\frac{\beta_2-\beta_1}{2}+\mathrm{i}\bar{\sigma}^2\tanh\frac{\beta_1-\beta_2}{2}$$
$$\times\exp\{-\mathrm{i}mr[\sinh(\beta_1+\mathrm{i}\theta)+\sinh(\beta_2+\mathrm{i}\theta)]\}\mathcal{A}_4(r,\beta_1+\mathrm{i}\theta,\beta_2+\mathrm{i}\theta)$$
$$+\mathrm{i}\bar{\sigma}^2\tanh\frac{\beta_1-\beta_2}{2}\mathcal{T}_2(r,\beta_1+\mathrm{i}\theta,\beta_2+\mathrm{i}\theta), \tag{B.13}$$

where

$$\mathcal{T}_2(r,\beta_1,\beta_2)=\mathcal{B}_2(r,\beta_1,\beta_2)+\mathcal{U}_2(r,\beta_1,\beta_2)+\mathcal{V}_2(r,\beta_1,\beta_2)+\mathcal{A}_4(r,\beta_1,\beta_2),$$

and

$$\mathcal{B}_2(r,\beta_1,\beta_2)=-\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}mr(\sinh\eta_1+\sinh\eta_2)}\coth\frac{\eta_1-\beta_1}{2}\coth\frac{\eta_2-\beta_2}{2}\tanh\frac{\eta_1-\eta_2}{2},$$
$$\mathcal{U}_2(r,\beta_1,\beta_2)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_2}{2\pi}\,\mathrm{e}^{\mathrm{i}mr(\sinh\beta_1+\sinh\eta_2)}\coth\frac{\eta_2-\beta_2}{2}\tanh\frac{\beta_1-\eta_2}{2},$$
$$\mathcal{V}_2(r,\beta_1,\beta_2)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1}{2\pi}\,\mathrm{e}^{\mathrm{i}mr(\sinh\eta_1+\sinh\beta_2)}\coth\frac{\eta_1-\beta_1}{2}\tanh\frac{\eta_1-\beta_2}{2},$$
$$\mathcal{A}_4(r,\beta_1,\beta_2)=-\mathrm{i}\,\mathrm{e}^{\mathrm{i}mr(\sinh\beta_1+\sinh\beta_2)}\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}mr(\sinh\eta_1+\sinh\eta_2)}$$
$$\cdot\tanh\frac{\eta_1-\beta_1}{2}\tanh\frac{\eta_2-\beta_2}{2}\tanh\frac{\eta_1-\eta_2}{2}.$$

Here, we again suppose that $0<\mathrm{Im}\beta_j<\pi$ for $j=1,2$.

Note that

$$\mathcal{T}_2(r,\beta_1+\mathrm{i}\pi,\beta_2+\mathrm{i}\pi)=\exp[-\mathrm{i}mr(\sinh\beta_1+\sinh\beta_2)]\mathcal{T}_2(r,\beta_1,\beta_2).$$

The two former terms in (B.13) contribute to $\Delta G(\beta_2,\beta_1|\beta_1,\beta_2)$, while all the rest of the terms in (B.13) should be assigned to $\delta G(\beta_2,\beta_1|\beta_1,\beta_2)$. Thus, the irreducible part of the two-particle matrix element takes the form

$$G_{\mathrm{irr}}(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)\equiv G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)-\Delta G(r,\theta;\beta_2,\beta_1|\beta_1,\beta_2)$$
$$=\left[\frac{G(\beta_1|\beta_1)G(\beta_2|\beta_2)}{G}-\frac{S(\beta_1|\beta_1)S(\beta_2|\beta_2)}{\bar{\sigma}^2}\right]$$
$$-\left[\frac{G(\beta_1|\beta_2)G(\beta_2|\beta_1)}{G}-\frac{[R(\beta_1|\beta_2)]^2}{\bar{\sigma}^2}\right]$$
$$-\left[\frac{1}{G}\left(\frac{G(\beta_1,\beta_2)}{E(\beta_1)E(\beta_2)}\right)^2+C_2(\beta_1,\beta_2)\right], \tag{B.14}$$

where

$$C_2(\beta_1,\beta_2)\equiv C_2(r,\theta;\beta_1,\beta_2)=2\bar{\sigma}^2\tanh^2\frac{\beta_2-\beta_1}{2}+2\bar{\sigma}^2\tanh^2\frac{\beta_1-\beta_2}{2}$$
$$\times\cos\{rm[\sinh(\beta_1+\mathrm{i}\theta)+\sinh(\beta_2+\mathrm{i}\theta)]\}[1-f^{(2)}(mr)]$$
$$-2\mathrm{i}\bar{\sigma}^2\tanh\frac{\beta_1-\beta_2}{2}\{1+\exp\{-\mathrm{i}mr[\sinh(\beta_1+\mathrm{i}\theta)+\sinh(\beta_2+\mathrm{i}\theta)]\}\}$$
$$\cdot\mathcal{T}_2(r,\beta_1+\mathrm{i}\theta,\beta_2+\mathrm{i}\theta).$$

Integration of (B.14) in r and θ gives the irreducible part of the factor $W(\beta_1, \beta_2)$:

$$W_{\mathrm{irr}}(\beta_1, \beta_2) = \int_0^\infty r\,\mathrm{d}r \int_0^\pi \mathrm{d}\theta\, G_{\mathrm{irr}}(r, \theta; \beta_2, \beta_1|\beta_1, \beta_2). \tag{B.15}$$

The integral in r here is convergent for small enough $|\beta_1 - \beta_2|$, and

$$W_{\mathrm{irr}}(\beta_1, \beta_2) = W_{\mathrm{irr}}(\beta_1 + \beta, \beta_2 + \beta)$$

for arbitrary β.

On the other hand, the integral

$$\int_0^\infty r\,\mathrm{d}r \int_0^\pi \mathrm{d}\theta\, \delta G(r, \theta; \beta_2, \beta_1|\beta_1, \beta_2)$$

converges and vanishes in the infinite momentum frame.

So, the local multi-quark contribution to the third-order meson mass correction takes the form

$$\delta_3 M_n^2 = -\left(\frac{h\bar{\sigma}}{m^2}\right)^3 \frac{8m^6}{M_n^2 - 4m^2} \frac{W_{\mathrm{irr}}(\beta_1, \beta_2)}{\bar{\sigma}^2}, \tag{B.16}$$

with $W_{\mathrm{irr}}(\beta_1, \beta_2)$ being given by (B.15). Three other third-order contributions to δM_n^2 come from the two-fermion Bethe–Salpeter equation in the infinite momentum frame (13) and from the quark mass and string tension renormalizations.

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